Math/Science Matters: Resource Booklets on Research in Math and Science Learning

Booklet 1
Cognitive Issues that Affect Math and Science Learning

Understanding Counts!: Teaching for Depth in Math and Science

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Summary Points

Why is it important to teach less, better?

- There are vast amounts of information.
- Time is limited.
- Students often can't remember, understand or use what they supposedly have learned.

How is deep understanding different from superficial knowledge?

- It focuses on meaningful patterns, related to structure and purpose.
- Fewer misconceptions result.
- Learned information is more connected.
- Information is more likely to be remembered, used, and applied to new areas.

What does research tell us about "teaching for deep understanding" in math and science as compared to teaching for broad coverage?

The question of what to teach and how to teach it is not new. So then, why is there so much interest right now in whether it is better to teach for deep understanding or for broad coverage? Some key issues fuel this interest:

❖ There are vast amounts of information to be learned;

We are in the middle of an information explosion. It simply isn't feasible to "cover" human knowledge. Even if our population can continue to share broad areas of common knowledge, we need to be selective about what those areas are.

❖ Students often can't remember, understand, or use what they supposedly have learned;

Educators are increasingly concerned with students' inability to use what they presumably have learned in school.[i] Too often, students don't remember what they've learned or don't understand it well enough to apply it outside of school.

❖ It takes time to develop deep understanding;

Developing deep understanding is time-consuming. Learners need time to explore concepts and grapple with the puzzles involved in understanding. Often students need to revisit concepts in different ways and to investigate different applications for the concept.

There is a tension precipitated by limited time, large amounts of information to be learned, and the goal of learning the information well. The goal of getting students to understand

topics deeply is pitted against coverage of material. What are the trade-offs between the two? If we teach for deeper understanding, can we expect a more educated population or will it be one with isolated areas of expertise and broad areas of unfamiliarity and ignorance? What does the available research suggest educators should do?

Education reform articles suggest that we should teach less, better -- that broad, superficial coverage should be replaced with more in-depth coverage of fewer areas with deep understanding as a goal.[ii] This argument hinges on at least two key points. The first is that deep understanding is beneficial -- different from and better than superficial knowledge. The second is that learners will achieve more than a few isolated areas of deep understanding -- that somehow what they learn will be useful beyond the contexts in which they learned it. Let's examine each of these issues in turn.

What is meant by "deep understanding" and how is it beneficial?

Those who study cognition use the term "deep understanding" to refer to how concepts are represented and connected by learners. They assume that learners create mental models -- pictures or images of concepts in their minds. These images can be connected -- or networked -- to other information that the learner has amassed. The way the images are networked represents the sense that the learner has made. The nature of the learning experience influences how learners construct the representations.[iii] Learners can connect these representations to produce larger networks of knowledge. Deep understanding implies that the information is well-represented and well-connected. The greater the number and strength of the connections, the deeper the understanding. New information can be well-connected to existing knowledge and/or the pieces of the new information can be well-connected from within.

Is deep understanding different from superficial knowledge? By definition, one expects it to be different because it describes well-connected networks of knowledge. But does a thinker with deep understanding *behave* differently than one with superficial understanding? According to researchers, the answer is "yes." They compared the behavior of experts -- people with deep understanding of a given area -- to that of novices -- people who are just becoming acquainted with the area. It turns out that experts structure their understanding and construe meaning from information differently than beginners.

Experts notice and attach importance to different patterns.[iv] They see the bigger picture -- perceiving larger, meaningful patterns and they reflectively monitor their activity helping them to regulate their attention.[v] For instance, a novice looking at fossil evidence might be drawn to the shape and size of the fossilized image while an expert would be more likely to notice the surrounding context in addition to the image -- noticing what type of minerals invaded the softer materials, where in the rock layer the fossil was found, where the fossil was found relative to other fossilized remains. The expert would be likely to reflect on whether the evidence found supported a commonly accepted theory and whether he or she had considered the right set of variables to know. The expert attends to the information differently and interprets it differently. Understanding something in depth is a different experience than understanding it in a superficial way.

So deep understanding is different, but does that make it *better* than superficial knowledge? Let's contrast deep understanding with the type of learning that often results from broad, superficial coverage. All too often, attempts to cover large amounts of information focus on rote learning -- attempts to commit to memory rather than take time to achieve understanding. Broad, superficial coverage affords less opportunity to explore concepts and discover the connections between them. The resulting knowledge is not connected in a relational manner.[vi] This increases the likelihood that errors will be made.[vii] For example, third graders with rote knowledge of multi-digit subtraction made more errors than those with conceptual understanding.[viii] "Meaningful learning arises not from the formation of connection through rote memorization but rather from a deep understanding of the structural relations in the problem."[ix]

Experts' understanding of the deeper structure of the information allows them to process patterns more efficiently and more effectively. They can monitor their own problem-solving and are faster overall at solving problems because they routinize their problem-solving to some extent which acts as a time-saving device.

Learners who understand something deeply are able to demonstrate this through "performances of understanding," including explaining, coming up with examples, applying, justifying, and so on.[x] Such performances not only show understanding, they help to *build it*. Through engaging in performances such as explanation, exemplification and so forth, learners are likely to discover misunderstandings and to work through them.

Making sense of the information actively engages learners in more complex knowledge construction.[xi] They must consider how the information relates to earlier knowledge. They are more likely to understand the design of the knowledge, how it is structured and the purpose for the particular structure.[xii] In contrast, superficial learning tends to reinforce insidious habits of mindlessness because there is a push to cover large amounts of material with little opportunity to explore the ideas.[xiii] Students adopt a passive stance and stop asking the questions which would have helped them pursue understanding.

Deep understanding increases the likelihood that students will be able to use the information outside of school. Superficial learning often remains inert.[xiv] It is present but students don't apply it to situations beyond school tests because they may not know how to apply it or they may not recognize opportunities to do so. When learners understand information at a structural level they are more likely to perceive ways to connect and use the information outside of school.

Research also shows that deep understanding also benefits learners by increasing their ability to store and access information.[xv] Deep understanding promotes remembering in two related ways:

1) The information can be stored in larger, more meaningful chunks, thus reducing the number of discrete pieces which need to be remembered;

> Finding meaningful relationships between pieces of information reduces the amount to be remembered.[xvi] The information is regrouped into larger, tightly connected units. Procedures are stored

and used more efficiently.[xvii] These chunks are more readily retrieved than a series of disconnected pieces. Research demonstrates that the depth with which information is processed - paying attention to superficial details as compared to making meaningful connections - affects recall more than the amount of time actually spent studying it.[xviii] Information that is assembled into a scheme of some sort - a picture in one's head, for example - is more easily recalled than disorganized information.[xix]

2) The connectedness to other topics results in more paths with which to access the information;

Memory research shows that when meaningful "chunks" of information are well connected to other understandings, there are more possible routes to recall it.[xx] For example, if a student relates a science lesson on friction to knowledge of what size bicycle tires to choose in certain conditions and/or what happens when playing on the playground slide, there are more contexts with which to recall the information. Thus, the information is less likely to deteriorate or decay. It is important that the networks are built as learning is taking place rather than accumulating large stores of information to be connected at some later time given the likelihood of greater decay of disconnected information.[xxi]

Deep understanding is clearly beneficial in a number of ways.

On balance, it is important to note that the benefits of deep understanding depend upon the particular skill or subject in question. Some skills may be enacted at a level where deep understanding does not affect performance. For example, there are some driving skills that are not improved by a deep understanding of how a car works. And there are certainly situations where superficial knowledge is adequate to serve the purpose at hand. For example, superficial knowledge of price scanners is adequate for a cashier ringing up groceries in the supermarket.

It is also important to note that increased understanding does not always immediately result in improved performance.[xxii] Children's behavior may temporarily regress as deeper understanding develops. For example, in language development as children shift from imitation to rule usage, they may switch from saying "went" to "go-ed." Previously, they appeared to understand the rules and the exceptions to the rules for creating past tense when they were merely imitating. As they actually learn the rules, their behavior may regress until the exceptions to the rules are also learned.[xxiii]

Can deep understanding in one area be helpful to understanding in other areas?

It appears that deep understanding is better than superficial knowledge in topic areas actually focused on. But what about those areas that aren't taught? Teaching for deep understanding is time-consuming and forces us to teach less content. Will learners emerge with broad areas of ignorance? This issue raises a serious challenge to teaching for deep understanding.

The crux of the issue rests on the question of whether deep understanding gained in one area can be helpful to learning in other areas. This question is fundamental to the process of education and has a long history of research surrounding it.[xxiv] Clearly, some ability to use previous learning in new situations is possible or each variation on experience that we confront would require training. However, to what extent are we able to apply previous learning to new areas and what is required to enable us

to do so?

Two assertions are often made about deep understanding in answer to these questions.[xxv] The first is that deep understanding enhances *transfer*. Transfer refers to students' ability to use the information in new contexts. The second assertion is that understanding is *generative*. Generativity refers to the likelihood that the information will engender new understandings. Deep understanding in one area is expected to help students not just apply, but to *construct* understandings in other areas. What does the current research say about each of these assertions? Are they warranted? A brief consideration of each follows.[xxvi]

Research shows that without deep understanding, it can be a challenge to transfer knowledge to new situations! Knowledge is often bound to the context in which it was initially learned and learners tend to compartmentalize it.[xxvii] Therefore, they may not recognize opportunities to transfer knowledge to new situations. When learners do try to apply information to new situations, they sometimes get caught up in surface details and miss underlying structural similarities.[xxviii] They miss the essence of what is to be transferred -- the deep relationship! Teaching for deep understanding addresses these problems because children understand the deep level structural relationships in learned content. This makes it easier to see the important relationships and increases the likelihood that they will recognize and apply the relationships elsewhere.

This is not to imply that transfer of deep understanding is an automatic process. It needs to be promoted.[xxix] Research shows that attention to "shepherding" transfer[xxx] helps students make connections. Students need help noticing instances where the deep level relationship exists.[xxxi] They benefit from guidance mapping it onto the new area.[xxxii] Learners also need to see how applying the relationship benefits them in other situations.[xxxiii]

What about generativity? Can understanding in one area help to create understanding in other areas? Deep understanding can be generative in ways that superficial knowledge cannot. For instance, research in mathematics suggests that when children hold networks of related concepts, they are more likely to create their own useful procedures and strategies.[xxxiv] On the other hand, when learning by rote in a superficial manner, children tend to create procedures and strategies that are not meaningful from an "expert" standpoint -- they do not represent relevant patterns. The procedures may be flawed[xxxv] or children may apply formulas in ways that don't make sense.[xxxvi] For example, when the number, "two hundred and three" is read, a child without deep conceptual understanding might write "2003" instead of 203.

Whether deep understanding is generative depends in part on what it is. Certain understandings help students make connections widely. For example, a deep understanding of gravity helps students to comprehend a range of events -- such as why certain planets have the type of atmosphere that they do, why astronauts need to exercise so much during space travel, what a snowball fight in space would look like,[xxxvii] or why our moon stays in its orbit. Certain understandings help students grasp other domains -- providing windows of insight beyond the particular cases studied. For example, deep understanding of the balance of life in the rain forest helps to generate understanding of the balance of life in the oceans. These types of understandings are generative. Other understandings are fun or

practical in specialized contexts without helping the learner much in other areas.[xxxviii] Learning chess moves or mastering a new computer language helps learners in specific areas without generating understanding elsewhere.

What does the research recommend that educators do when it comes to the trade-offs between teaching for deep understanding versus broad coverage?

In summary, the research makes a strong case for teaching for deeper understanding. It benefits students by increasing the likelihood that they'll:

- remember what they have learned.
- understand the design of the knowledge at a structural and purposeful level.
- use their understanding beyond the contexts of school.
- see greater connectedness in their learning.
- generate new, useful insights from what they have learned.

Teachers need to pay particular attention to transfer and generativity. They need to foster students' ability to make connections and apply understandings to new areas. They should chose generative topics for deep investigation. These topics should share important analogies with other topics of interest - shedding light on central issues and enabling varied extrapolation and connection-making.[xxxix] The topics should invite many ways to demonstrate understanding and provide for assessments that measure knowledge with depth.

Students also need to learn "finding out" tools so that they can get specific information when they need it. For instance, students may only learn deeply about one type of ecosystem in their formal schooling, but they need to know how to research information about others in the event that the information is needed. Learning to find out is crucial to helping students to continue to learn throughout their lives - especially important in our information age.

In summary, this review suggests that educators should:

- teach for deeper understanding.
- use strategies to help students transfer understandings.
- seek out the most generative topics for in-depth exploration.
- teach "finding out" skills to help students learn throughout their lives.

This should foster learning that endures, is useful in the real world, and produces the most educated and enabled populace.

For Further Information

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v. Glaser, R. (1992). Expert knowledge and the processes of thinking. In D. F. Halpern (Ed.), *Enhancing thinking skills in the sciences and mathematics* (pp. 63-75). Hillsdale, NJ: LEA.

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ix. Singley, M. K., & Anderson, J. R. (1989). *The transfer of cognitive skill*. p. 9. Cambridge, MA: Harvard University Press.

x. Perkins, 1992.

xi. King, A. (1994). Guiding knowledge construction in the classroom: Effects of teaching children how to question and how to explain. *American Educational Research Journal*, 31(2), 338-368.

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xiii. Newman, F. M (1988, January). Can depth replace coverage in the high school curriculum? *Phi Delta Kappan*, 345-348.

xiv. Perkins, 1992.

xv. See Hiebert & Carpenter, 1992 for a critique.

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xviii. Craik, F. I. M. & Tulving, E. (1975). Depth of processing and the retention of words in episodic memory. *Journal of Experimental Psychology*, 104, 268-294.

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xxi. Nolan, J. D. (1973). Conceptual and rote learning in children. *Teacher's College Record*, 75, 251-258.

xxii. e.g. Hiebert & Carpenter, 1992.
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xxiv. This issue is at the heart of the entire process of education and our ability to successfully "educate" learners in an information age. See Singley & Anderson, 1989 for a review.

xxv. Hiebert & Carpenter, 1992.

xxvi. The interested reader is referred to Hiebert & Carpenter, 1992 for an extensive review.

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